IUCAA GR Refresher Course Tutorials by BM

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Tutorial:5 (27-06-2025)

Problem: Deriving the First-Order Trajectory Equation in Schwarzschild Spacetime Using Killing Vectors

The Schwarzschild metric in standard coordinates (t, r, θ, ϕ) is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Due to the spherical symmetry, one can restrict the motion of a test particle to the **equatorial plane** $(\theta = \frac{\pi}{2})$, without loss of generality.

<u>**Tasks:**</u>

- Identify the two **Killing vectors** of the Schwarzschild spacetime corresponding to time translation and axial symmetry:
  $\xi^\mu_{(t)} = \partial_t$, $\xi^\mu_{(\phi)} = \partial_\phi$.
- Use these Killing vectors to find the conserved quantities along the particle's geodesic:

  Energy per unit mass:

  $$E = -g_{\mu\nu}\xi^\mu_{(t)}u^\nu = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}$$

  Angular momentum per unit mass:

  $$L = g_{\mu\nu}\xi^\mu_{(\phi)}u^\nu = r^2\frac{d\phi}{d\tau}$$

- Using the normalization condition for timelike geodesics:

$$g_{\mu\nu} u^\mu u^\nu = -1,$$

derive a **first-order differential equation** for the radial trajectory $\left(\frac{dr}{d\tau}\right)^2$ in terms of E and L.

- **Optional:** Convert the result to an orbital equation in terms of $u = 1/r$ and ϕ.

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This question develops the skill of using spacetime symmetries (Killing vectors) to reduce the geodesic equations to a manageable first-order form, and is a foundational technique for analyzing orbits and gravitational lensing in Schwarzschild geometry.